Perhaps the prettiest number system of all is the *balanced ternary* notation, which consists of radix-3 representation using -1, 0, and $+1$ as "trits" (ternary digits) instead of 0, 1, and 2. If we let the symbol $\bar{1}$ stand for -1, we have the following examples of balanced ternary numbers:

Balanced ternary	Decimal
$10\bar{1}$	8
$11\bar{1}0.\bar{1}\,\bar{1}$	$32\frac{5}{9}$
$\bar{1}\,\bar{1}10.11$	$-32\frac{5}{9}$
$\bar{1}\,\bar{1}10$	-33
$0.11111\ldots$	$\frac{1}{2}$

One way to find the representation of a number in the balanced ternary system is to start by representing it in ordinary ternary notation; for example,

$$208.3 = (21201.022002200220\ldots)_3.$$

(A very simple pencil-and-paper method for converting to ternary notation is given in exercise 4.4–12.) Now add the infinite number $\ldots11111.11111\ldots$ in ternary notation; we obtain, in the example above, the infinite number

$$(\ldots11111210012.210121012101\ldots)_3.$$

Finally, subtract $\ldots11111.11111\ldots$ by decrementing each digit; we get

$$208.3 = (10\bar{1}\bar{1}01.10\bar{1}010\bar{1}010\bar{1}0\ldots)_3. \tag{8}$$

This process may clearly be made rigorous if we replace the artificial infinite number $\ldots11111.11111\ldots$ by a number with suitably many ones.

The balanced ternary number system has many pleasant properties:

a) The negative of a number is obtained by interchanging 1 and $\bar{1}$.

b) The sign of a number is given by its most significant nonzero trit, and in general we can compare any two numbers by reading them from left to right and using lexicographic order, as in the decimal system.

c) The operation of rounding to the nearest integer is identical to truncation; in other words, we simply delete everything to the right of the radix point.

Addition in the balanced ternary system is quite simple, using the table

	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}\,\bar{1}$	$\bar{1}$	0
0	$\bar{1}$	0	1
1	0	1	$1\bar{1}$

	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}$	0	1
0	0	1	$1\bar{1}$
1	1	$1\bar{1}$	10

(The three inputs to the addition are the digits of the numbers to be added and the carry digit.) Subtraction is negation followed by addition. Multiplication

also reduces to negation and addition, as in the following example:

$$\begin{array}{rrrrrrrrl}
 & & & 1 & \bar{1} & 0 & \bar{1} & & [17] \\
 & & & 1 & \bar{1} & 0 & \bar{1} & & [17] \\
\hline
 & & & \bar{1} & 1 & 0 & 1 & & \\
 & & \bar{1} & 1 & 0 & 1 & & & \\
 & 1 & \bar{1} & 0 & \bar{1} & & & & \\
\hline
 & 0 & 1 & 1 & \bar{1} & \bar{1} & 0 & 1 & [289]
\end{array}$$

Representation of numbers in the balanced ternary system is implicitly present in a famous mathematical puzzle, commonly called "Bachet's problem of weights" although it was already stated by Fibonacci four centuries before Bachet wrote his book, and by Ṭabarī in Persia more than 100 years before Fibonacci. [See W. Ahrens, *Mathematische Unterhaltungen und Spiele* **1** (Leipzig: Teubner, 1910), Section 3.4; H. Hermelink, *Janus* **65** (1978), 105–117.] Positional number systems with negative digits were invented by J. Colson [*Philos. Trans.* **34** (1726), 161–173], then forgotten and rediscovered about 100 years later by Sir John Leslie [*The Philosophy of Arithmetic* (Edinburgh: 1817); see pages 33–34, 54, 64–65, 117, 150], and A. Cauchy [*Comptes Rendus Acad. Sci. Paris* **11** (1840), 789–798]. Cauchy pointed out that negative digits make it unnecessary for a person to memorize the multiplication table past 5×5. A claim that such number systems were known in India long ago [J. Bharati, *Vedic Mathematics* (Delhi: Motilal Banarsidass, 1965)] has been refuted by K. S. Shukla [*Mathematical Education* **5**, 3 (1989), 129–133]. The first true appearance of "pure" balanced ternary notation was in an article by Léon Lalanne [*Comptes Rendus Acad. Sci. Paris* **11** (1840), 903–905], who was a designer of mechanical devices for arithmetic. The system was mentioned only rarely for 100 years after Lalanne's paper, until the development of the first electronic computers at the Moore School of Electrical Engineering in 1945–1946; at that time it was given serious consideration along with the binary system as a possible replacement for the decimal system. The complexity of arithmetic circuitry for balanced ternary arithmetic is not much greater than it is for the binary system, and a given number requires only $\ln 2 / \ln 3 \approx 63\%$ as many digit positions for its representation. Discussions of the balanced ternary number system appear in *AMM* **57** (1950), 90–93, and in *High-speed Computing Devices*, Engineering Research Associates (McGraw–Hill, 1950), 287–289. The experimental Russian computer SETUN was based on balanced ternary notation [see *CACM* **3** (1960), 149–150], and perhaps the symmetric properties and simple arithmetic of this number system will prove to be quite important some day — when the "flip-flop" is replaced by a "flip-flap-flop."

Positional notation generalizes in another important way to a *mixed-radix* system. Given a sequence of numbers $\langle b_n \rangle$ (where n may be negative), we define

$$\begin{bmatrix} \ldots, a_3, a_2, a_1, a_0; a_{-1}, a_{-2}, \ldots \\ \ldots, b_3, b_2, b_1, b_0; b_{-1}, b_{-2}, \ldots \end{bmatrix} = \cdots + a_3 b_2 b_1 b_0 + a_2 b_1 b_0 + a_1 b_0 + a_0 + a_{-1}/b_{-1} + a_{-2}/b_{-1} b_{-2} + \cdots . \tag{9}$$