Application of Ternary Algebra to the Study of Static Hazards*

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Abstract. This paper is concerned with the study of static hazards in combinational switching circuits by means of a suitable ternary switching algebra. Techniques for hazard detection and elimination are developed which are analogous to the Huffman-McCluskey procedures. However, gate and series-parallel contact networks are treated by algebraic methods exclusively, whereas a topological approach is applied to non-series-parallel contact networks only. Moreover, the paper derives necessary and sufficient conditions for a ternary function to adequately describe the steady-state and static hazard behavior of a combinational network. The sufficiency of these conditions is proved constructively leading to a method for the synthesis of combinational networks containing static hazards as specified. The section on non-series-parallel contact networks also includes a brief discussion of the applicability of lattice matrix theory to hazard detection. Finally, hazard prevention in contact networks by suitable contact sequencing techniques is discussed and a ternary map method for the synthesis of such networks is explained.

1. *Introduction*

The theory of static hazards in binary switching circuits has been established by Huffman [1] and McCluskey [2]. This theory leads to methods of detecting hazards and to design procedures for hazard-free circuits. Both Huffman [1] and McCluskey [2] develop the theory of static hazards within the mathematical framework of Boolean algebra.

Metze [3] has discussed the application of a four-valued algebra to hazard detection in electronic switching circuits, whereas a number of authors [4, 5, 6] have suggested the use of ternary logic to the study of static hazards. It is the purpose of this paper to take up these suggestions and to establish a precise mathematical theory of both the steady-state and the transient behavior of combinational switching circuits, based on a suitable ternary algebra.

The basic techniques for hazard detection and elimination to be derived in this paper will be analogous to the Huffman-McCluskey procedures.

However, as far as gate (and series-parallel contact) networks are concerned the methods used will be entirely algebraic. Moreover, some additional results and methods concerning static hazards will be derived.

* Received August, 1963. The research reported in this paper has been sponsored in part by the Air Force Office of Scientific Research, OAR through the European Office, Aerospace Research, United States Air Force. Part of this paper is based on a thesis for the degree of Master of Science submitted by Shlomo Rinon to the Senate of the Technion, Israel Institute of Technology, Haifa, Israel, July 1963.

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Journal of the Association for Computing Machinery, Vol. 11, No. 1 (January, 1964), pp. 84–97

2. *A Ternary Switching Algebra*

The ternary algebra to be developed is based on the set $L = \{0, \frac{1}{2}, 1\}$. In the sequel the values 0 and 1 will be interpreted as usual in conventional switching theory [7, 8]. The value $\frac{1}{2}$ will be interpreted to represent transient states or positions of binary switching elements or, alternatively, uncertain values of binary switching variables, occuring during transition periods of the combinational switching circuit under consideration.

The Boolean operations of addition $(+)$, multiplication $(\cdot)$, and inversion $(-)$ are extended to L as follows:

$+$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

$\cdot$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

x	0	$\frac{1}{2}$	1
$\bar{x}$	1	$\frac{1}{2}$	0

Thus

$$x + y = \max(x, y), \quad x \cdot y = \min(x, y), \quad \bar{x} = 1 - x; x, y \text{ in } L.$$

These ternary operations properly describe both the steady-state and the transient behavior of the corresponding binary basic logic circuits.

Evidently the system $(L, +, \cdot)$ forms a distributive lattice [9] with zero element 0 and universal element 1. Thus the following laws hold for any x, y, $z \in L$:

Idempotent:	$x + x = x$	$x \cdot x = x$
Commutative:	$x + y = y + x$	$x \cdot y = y \cdot x$
Associative:	$(x + y) + z = x + (y + z)$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
Absorption:	$x + x \cdot y = x$	$x \cdot (x + y) = x$
Distributive:	$(x + y) \cdot z = x \cdot z + y \cdot z$	$x \cdot y + z = (x + z) \cdot (y + z)$

The elements 0, 1 satisfy for every $x \in L$:

$$x \cdot 0 = 0 \quad x \cdot 1 = x \qquad x + 0 = x \quad x + 1 = 1$$

The rules of de Morgan also hold:

$$\overline{x + y} = \bar{x} \cdot \bar{y} \qquad \overline{x \cdot y} = \bar{x} + \bar{y}$$

Furthermore, $\bar{\bar{x}} = x$.

3. *B-Ternary Switching Functions*

Let n be a positive integer and let L^n denote the nth Cartesian power of L, i.e. L^n is the set of all ternary vectors $(x_1, \cdots, x_n)$, $x_i \in L$, $i = 1, \cdots, n$.

A *ternary n-variable function* is a function on L^n to L. We now define a *B-ternary n-variable function* $f(x_1, \cdots, x_n)$ as any ternary n-variable function which is either constantly 0 or 1, or obtained from its arguments $x_1, \cdots, x_n$ by successive

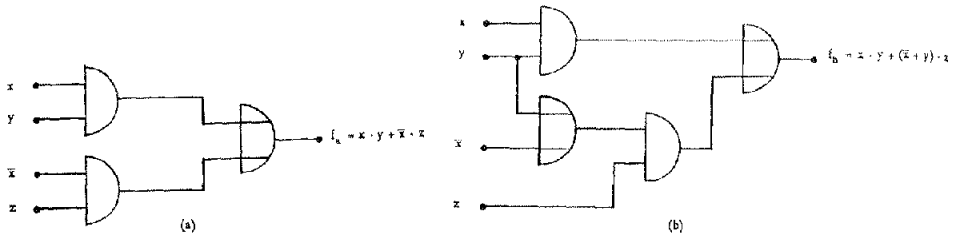


FIG. 1. Gate networks with same steady-state but different transient behavior

TABLE 1. PARTIAL TABLES OF COMBINATIONS FOR FIGURE 1

x	y	z	f_a	f_b
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	0	0
1	1	0	1	1
1	1	1	1	1
$\frac{1}{2}$	0	0	0	0
$\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	1	1	$\frac{1}{2}$	1

application of the ternary operations of addition (+), multiplication ($\cdot$), and inversion ($-$).

It follows that the overall performance (including transient behavior) of a binary electronic combinational switching circuit composed of AND-, OR- and NOT-logic circuits will be adequately described by the corresponding B-ternary function.

The analogous statement holds for series-parallel contact networks. The *B-ternary transmission function* of a two-terminal series-parallel contact network is obtained as usual by associating addition and multiplication with paralleling and cascading, respectively. A normally open x_i-contact is represented by the (ternary) variable x_i, a normally closed x_i-contact by $\bar{x}_i$.

Let us illustrate the situation by suitable examples. Consider the two gate networks shown in Figure 1. They are clearly equivalent, as far as steady-state behavior is concerned. However, their B-ternary functions are different. Namely, the ternary output function of the network of Figure 1(a) is $f_a = x \cdot y + \bar{x} \cdot z$, whereas the network of Figure 1(b) has the output function $f_b = x \cdot y + (\bar{x} + y) \cdot z$. A 3-variable ternary function may be completely described by a 27-row table of combinations. Partial combination tables for f_a and f_b are shown in Table 1.

Note that for the input combinations (0, 1, 1) and (1, 1, 1) both networks yield output 1, whereas $f_a(\frac{1}{2}, 1, 1) = \frac{1}{2}$ and $f_b(\frac{1}{2}, 1, 1) = 1$.

It follows that during a transition from input state (0, 1, 1) to (1, 1, 1), or

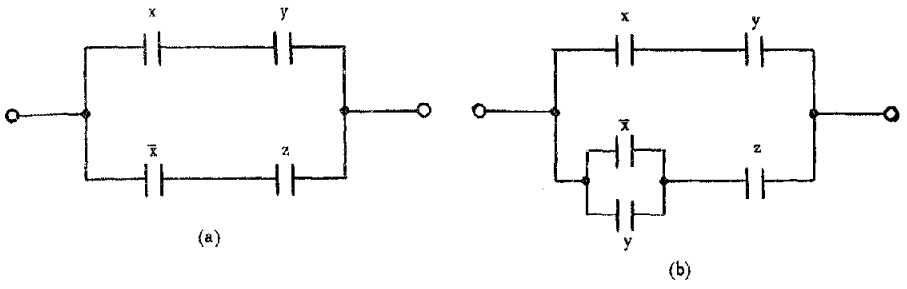


FIG. 2. Contact networks corresponding to the gate networks of Fig. 1

vice versa, the output of the network of Figure 1(b) definitely retains the value 1, whereas the behavior of the network of Figure 1(a) is uncertain and may actually yield a spurious 0 output. The importance of preventing such spurious outputs (hazard prevention) is assumed to be known [7].

Contact networks realizing the above B-ternary functions f_a and f_b are shown in Figure 2, (a) and (b), respectively. Hence it follows from Table 1, that during a transition from input state (0, 1, 1) to (1, 1, 1) or vice versa the transmission value of the network of Figure 2(b) will remain 1, whereas in the network of Figure 2(a) a momentary 0 transmission value may occur.

4. *Static Hazards*

Let $V = \{0, 1\} \cdot V^n (\subset L^n)$ is the set of all binary n-dimensional vectors. Two vectors of V^n, $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$, are *adjacent* if they differ in exactly one component, say $b = (a_1, \cdots, a_{i-1}, \bar{a}_i, a_{i+1}, \cdots, a_n)$.

We define the transition vector $a/b (\in L^n)$ as

$$a/b = (a_1, \cdots, a_{i-1}, \tfrac{1}{2}, a_{i+1}, \cdots, a_n).$$

In the sequel we shall need the following:

THEOREM 1. *Let $f(x_1, \cdots, x_n)$ be a B-ternary function and a, b any adjacent vectors of V^n. Then $f(a/b) \neq \overline{f(a)}$.*

Proof. Clearly any x_i, $i = 1, \cdots, n$, (considered as function of L^n to L) satisfies Theorem 1. Also, if f satisfies Theorem 1, so does $\bar{f}$. Assuming now that the B-ternary functions $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ satisfy Theorem 1, we show that this will also hold for $f + g$. Indeed, suppose that $(f + g)(a/b) = \overline{(f + g)(a)}$. Hence $f(a/b) + g(a/b) = \overline{f(a) + g(a)} = \overline{f(a)} \cdot \overline{g(a)}$.

We assume (without restricting the generality of our considerations) that

$$f(a/b) \geqq g(a/b). \tag{1}$$

Hence

$$f(a/b) + g(a/b) = f(a/b) = \overline{f(a)} \cdot \overline{g(a)} \in V. \tag{2}$$

Thus, we cannot have $f(a/b) = \frac{1}{2}$. If $f(a/b) = 1$, it follows from (2) that $f(a) = g(a) = 0$, i.e. $f(a/b) = \overline{f(a)}$. If $f(a/b) = 0$, (1) implies $g(a/b) = 0$. By (2), either $f(a) = 1$ or $g(a) = 1$. In both cases our assumption that f and

g satisfy Theorem 1 is contradicted. It follows that Theorem 1 also holds for $f + g$, and consequently for $f \cdot g = \overline{\bar{f} + \bar{g}}$. Theorem 1 is thus proved.

We now define a *V-ternary function* as a ternary function $f(x_1, \cdots, x_n)$ such that $f(a) \in V$ for every $a \in V^n$. A V-ternary function f *induces* a binary function F on V^n to V, determined by $F(a) = f(a)$ for every $a \in V^n$. Two V-ternary n-variable functions f and g inducing the same binary function F are *V-equivalent.*

Clearly every B-ternary function is also V-ternary, but the converse is not true. E.g. any nontrivial V-ternary function $f(x_1, \cdots, x_n)$ for which $f(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \neq \frac{1}{2}$ cannot be B-ternary. This follows immediately from the fact that $\frac{1}{2} + \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} = \bar{\frac{1}{2}} = \frac{1}{2}$. Other V-ternary functions which are not B-ternary may be specified by means of Theorem 1.

If the B-ternary function f describes the complete performance of a binary combinational network, its steady-state behavior will be represented by F, the binary function induced by f.

Let $f(x_1, \cdots, x_n)$ be a V-ternary function, and a, b adjacent vectors of V^n. The vector a/b is a *static hazard* of f if $f(a) = f(b) \neq f(a/b)$.

If $f(a) = f(b) = 1$, a/b is a *1-hazard*;
if $f(a) = f(b) = 0$, a/b is a *0-hazard*;
if f has no static hazards it is *hazard-free.*

If f is B-ternary and a/b a static hazard of f, Theorem 1 implies that $f(a/b) = \frac{1}{2}$.

5. *Hazard Detection*

Given a single-output gate network composed of AND-gates, OR-gates and inverters, or a two-terminal series-parallel contact network, the corresponding B-ternary function f may be directly read off, as has been illustrated by means of Figures 1 and 2.

A network is called *hazard-free,*[1] if its B-ternary function f is hazard-free. All static hazards of any V-ternary function $f(x_1, \cdots, x_n)$ may, of course, be detected directly from its table of combinations. Actually only part of this table is required. Namely, for every $a \in V^n$ and for every transition vector a/b, where a,b are adjacent vectors satisfying $f(a) = f(b)$.

The above method for the detection of static hazards is practicable only if the number of variables is small. We shall now develop hazard detection procedures suitable for a larger number of variables. Let $f(x_1, \cdots, x_n)$ be a B-ternary function. By successive applications of the algebraic rules given in Section 2 the function f may be expanded about say x_1 as follows:

$$f(x_1, \cdots, x_n) = x_1 \cdot f_1(x_2, \cdots, x_n) + \bar{x}_1 \cdot f_2(x_2, \cdots, x_n) + x_1 \cdot \bar{x}_1 \cdot f_3(x_2, \cdots, x_n) + f_4(x_2, \cdots, x_n), \tag{3}$$

where $f_1, \cdots, f_4$ are also B-ternary.

[1] This paper considers *static* hazards only. Thus, the term "hazard-free" is used instead of "free from static hazards." A network free from static hazards may, however, still have *dynamic* hazards. For details, the reader is referred to [1] and [2].

Now let $a = (a_1, \cdots, a_n)$ be a vector of V^n, and let $b = (\bar{a}_1, a_2, \cdots, a_n)$. The vector $a/b = (\frac{1}{2}, a_2, \cdots, a_n)$ is a 1-hazard of f if and only if $f(a) = f(b) = 1$ and $f(a/b) = \frac{1}{2}$. By (3) these conditions are equivalent to:

$$f_1(a_2, \cdots, a_n) = 1, \quad f_2(a_2, \cdots, a_n) = 1, \quad f_4(a_2, \cdots, a_n) = 0. \tag{4}$$

The values of $a_2, \cdots, a_n$ in (4) are 0 or 1. All binary vectors $(a_2, \cdots, a_n)$ satisfying (4) can therefore be obtained by solving the corresponding Boolean equations. Summarizing, we obtain

THEOREM 2. *The vector $(\frac{1}{2}, a_2, \cdots, a_n)$ is a 1-hazard of the B-ternary function $f(x_1, \cdots, x_n)$ given in (3) if and only if the binary vector $(a_2, \cdots, a_n)$ is a solution of the following Boolean equations*:

$$f_1(x_2, \cdots, x_n) = 1, \quad f_2(x_2, \cdots, x_n) = 1, \quad f_4(x_2, \cdots, x_n) = 0.$$

The expansion (3) is also useful for the determination of 0-hazards. The vector $a/b = (\frac{1}{2}, a_2, \cdots, a_n)$ is a 0-hazard of f if and only if $f(a) = f(b) = 0$ and $f(a/b) = \frac{1}{2}$. These conditions are equivalent to:

$$f_1(a_2, \cdots, a_n) = 0, \quad f_2(a_2, \cdots, a_n) = 0, \quad f_3(a_2, \cdots, a_n) = 1, \quad f_4(a_2, \cdots, a_n) = 0. \tag{5}$$

Hence we have the following:

THEOREM 3. *The vector $(\frac{1}{2}, a_2, \cdots, a_n)$ is a 0-hazard of the B-ternary function $f(x_1 \cdots, x_n)$ given in (3) if and only if the binary vector $(a_2, \cdots, a_n)$ is a solution of the following Boolean equations*:

$$f_1(x_2, \cdots, x_n) = 0, \quad f_2(x_2, \cdots, x_n) = 0, \quad f_3(x_3, \cdots, x_n) = 1, \quad f_4(x_2, \cdots, x_n) = 0.$$

To illustrate the application of Theorems 2 and 3 we consider the B-ternary function [2, p. 27],

$$f(w, x, y, z) = w \cdot x \cdot y + (w + z) \cdot (\bar{w} + \bar{y}).$$

Expanding f about w in accordance with (3) we obtain

$$f = w \cdot (x \cdot y + \bar{y}) + \bar{w} \cdot z + w \cdot \bar{w} \cdot 1 + \bar{y} \cdot z.$$

The Boolean equations corresponding to (4) become

$$xy + \bar{y} = x + \bar{y} = 1, \quad z = 1, \quad \bar{y}z = 0.$$

One immediately obtains the unique solution $z = 1, y = 1, x = 1$, leading to the 1-hazard $(\frac{1}{2}, 1, 1, 1)$. Similarly, the equations corresponding to (5) become:

$$x + \bar{y} = 0, \quad z = 0, \quad 1 = 1, \quad \bar{y}z = 0.$$

Again, we have a unique solution, namely $x = 0, y = 1, z = 0$. Thus, $(\frac{1}{2}, 0, 1, 0)$ is a 0-hazard.

From Theorems 2 and 3 one easily deduces

THEOREM 4. *Let F be a binary function (on V^n to V) and f the B-ternary func-*

tion corresponding to the representation of F as sum of all its prime implicants. Then f is hazard-free.

PROOF. By Theorem 3 it is evident that f has no 0-hazards. Referring to Theorem 2, assume that the first two equations of (4) are satisfied. Then there exists a prime implicant ϕ of f which does not include the variable x_1, such that $\phi(a_2, \cdots, a_n) = 1$. Hence $f_4(a_2, \cdots, a_n) = 1$; i.e. (4) is not satisfied. Thus f has no 1-hazards either.

Dually, a B-ternary function $f(x_1, \cdots, x_n)$ may be expanded as follows:

$$f(x_1, \cdots, x_n) = [x_1 + g_1(x_2, \cdots, x_n)]\cdot[\bar{x}_1 + g_2(x_2, \cdots, x_n)] \cdot[x_1 + \bar{x}_1 + g_3(x_2, \cdots, x_n)]\cdot g_4(x_2, \cdots, x_n). \tag{6}$$

This expansion leads to 0- and 1-hazard testing procedures dual to Theorems 2 and 3. We omit the details of these procedures. Theorems 2, 3 and their duals are analogous to the hazard testing methods derived by Huffman [1] and McCluskey [2]. Similarly, the dual of Theorem 4 is easily derived.

6. *Synthesis of Hazard-Free Networks*

Using the terminology introduced the problem of designing hazard-free combinational networks may be formulated as follows:

Synthesis Problem. Given a binary function F on V^n to V, find a hazard-free B-ternary function f inducing F.

Once such a function f has been found, the design of a gate network or series-parallel contact network corresponding to f is straightforward. The steady-state behavior of this network will be as specified by F, and its transient behavior will correspond to f, i.e. will be hazard-free.

Before proceeding with the discussion of the synthesis of hazard-free circuits, it is convenient to introduce the following auxiliary single-variable functions:

x	x^0	$x^{\frac{1}{2}}$	x^1
0	1	0	0
$\frac{1}{2}$	0	1	0
1	0	0	1

These functions play an important role in ternary logic [10] and ternary switching [11, 12]. One easily verifies the following *Expansion Theorem* for ternary functions:

$$f(x_1, \cdots, x_n) = \sum_{(c_1, \cdots, c_n)} f(c_1, \cdots, c_n)\cdot x_1^{c_1}\cdot x_2^{c_2}\cdot \cdots \cdot x_n^{c_n} \tag{7}$$

where the summation is over all ternary vectors $(c_1, \cdots, c_n)$ of L^n.

It has been shown by Huffman [1] and McCluskey [2] that the synthesis problem formulated above always has a solution. We now derive a similar result, namely the following:

THEOREM 5. *Let $f(x_1, \cdots, x_n)$ be a V-ternary function satisfying the condition $f(a/b) \neq \overline{f(a)}$ for every pair of adjacent vectors (a, b) of V^n. Then there exists a*

B-ternary function f_B, V-equivalent to f, such that $f_B(a/b) = f(a/b)$ for every pair of adjacent vectors (a, b) of V^n.

Proof. First consider the case $n = 1$. All V-ternary functions of a single variable $f(x)$ satisfying the condition of Theorem 5, are listed below:

x	0	$\frac{1}{2}$	1
$f_1(x)$	0	0	0
$f_2(x)$	0	$\frac{1}{2}$	0
$f_3(x)$	0	$\frac{1}{2}$	1
$f_4(x)$	1	$\frac{1}{2}$	0
$f_5(x)$	1	$\frac{1}{2}$	1
$f_6(x)$	1	1	1

All these functions are B-ternary. Namely $f_1 = 0$, $f_2 = x \cdot \bar{x}$, $f_3 = x$, $f_4 = \bar{x}$, $f_5 = x + \bar{x}$, $f_6 = 1$.

Now let $f(x_1, \cdots, x_n)$ be as specified in Theorem 5 and a, b adjacent vectors of V^n. To simplify the notations, let us assume that $a = (a_1, a_2, \cdots, a_n)$ and $b = (\bar{a}_1, a_2, \cdots, a_n)$. With (a, b) we associate a function $f_{ab}(x_1, \cdots, x_n)$ defined as follows:

$$f_{ab} = x_1^{a_1} \cdot x_2^{a_2} \cdot \cdots \cdot x_n^{a_n} \cdot f(a) + x_1^{\frac{1}{2}} \cdot x_2^{a_2} \cdot \cdots \cdot x_n^{a_n} f(a/b) + x_1^{\bar{a}_1} \cdot x_2^{a_2} \cdot \cdots \cdot x_n^{a_n} \cdot f(b). \tag{8}$$

Next, we define $g(x_1, \cdots, x_n) = \sum_{(a,b)} f_{ab}$, where the summation is to be taken over all adjacent vector pairs (a, b) of V^n. An immediate consequence of the Expansion Theorem (7) is that g is V-equivalent to f, and furthermore $g(a/b) = f(a/b)$ for every adjacent vector pair (a, b). Returning now to expression (8) we may write:

$$f_{ab} = [x_1^{a_1} \cdot f(a) + x_1^{\frac{1}{2}} \cdot f(a/b) + x_1^{\bar{a}_1} \cdot f(b)] \cdot x_2^{a_2} \cdot \cdots \cdot x_n^{a_n}. \tag{9}$$

Denoting the expression in the square bracket of (9) as g_{ab}, g_{ab} is a V-ternary function of the single variable x_1 satisfying the condition of Theorem 4. Hence g_{ab} is B-ternary. Let us now replace $x_2^{a_2}, \cdots, x_n^{a_n}$ in (9) by $x_2^{(a_2)}, \cdots, x_n^{(a_n)}$, where $x_i^{(0)}$ stands for $\bar{x}_i$ and $x_i^{(1)}$ for x_i. Let the resulting function be

$$f_{ab}^B = g_{ab} \cdot x_2^{(a_2)} \cdot \cdots \cdot x_n^{(a_n)}.$$

f_{ab}^B is B-ternary. Considering that $x_i^{(0)} = x_i^0 + \frac{1}{2} x_i^{\frac{1}{2}}$ and $x_i^{(1)} = x_i^1 + \frac{1}{2} x_i^{\frac{1}{2}}$, one easily verifies that the B-ternary function $f_B = \sum_{(a,b)} f_{ab}^B$ satisfies the requirements of Theorem 5.

Theorem 5 enables us to synthesize hazard-free networks with any specified steady-state behavior. Moreover, it enables us to design combinational networks with a specified steady-state behavior containing certain specified static hazards

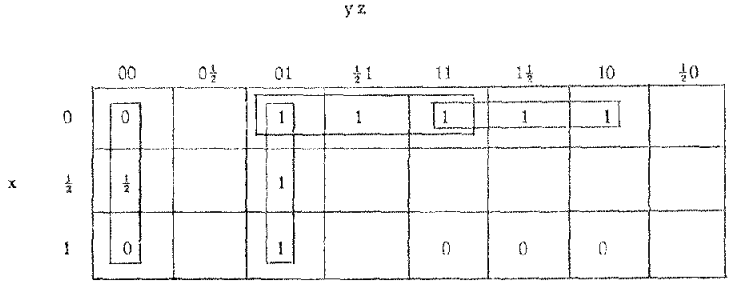


FIG. 3. Partial specification of ternary function f by ternary map

and only those specified.[2] The utilization of such circuits and methods for designing them have been discussed in a recent paper by Eichelberger [13], to which the reader is referred for further details.

To illustrate the synthesis procedure based on Theorem 5, let us consider a ternary function f partially specified by means of the (incomplete) ternary map of Figure 3. The f_{ab}-terms corresponding to the four cell triplets indicated in Figure 3 are as follows:

$$y^0z^0x\bar{x}, \quad y^0z^1, \quad x^0z^1, \quad x^0y^1.$$

Note that these triplets cover all cells with nonzero entries. It follows from the proof of Theorem 5 that the following B-ternary function will have the values specified in Figure 3:

$$f_B = \bar{y}\bar{z}x\bar{x} + \bar{y}z + \bar{x}z + \bar{x}y.$$

However, this function may be further simplified. E.g., by an easy adaptation of the binary map method the following simplification of f_B is obtained:

$$f_B' = x\bar{x} + \bar{y}z + \bar{x}z + \bar{x}y.$$

Techniques for obtaining minimal hazard-free sum-of-products or product-of-sums expressions have been described by McCluskey [2] and Eichelberger [13]. With Theorem 4 as starting point, a minimal circuit realizing F is obtained by a suitable selection of prime implicants of F, such that the B-ternary function g corresponding to their sum satisfies:

$$F(a) = 1 \Rightarrow g(a) = 1 \text{ for every } a \in V^n;$$

$$F(a) = F(b) = 1 \Rightarrow g(a/b) = 1 \text{ for every transition } a/b.$$

Similarly, the dual of Theorem 4 will lead to a minimal product-of-sums expression.

7. *Non-Series-Parallel Contact Networks*

The ternary algebra introduced also facilitates the study of hazards in non-series-parallel contact networks. Consider e.g. the two-terminal contact network

[2] In this respect Theorem 5 is somewhat more general than the corresponding results in [1] and [2].

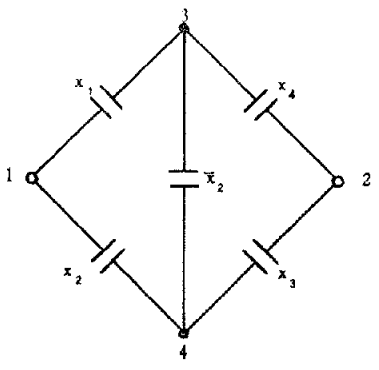


FIG. 4. A two-terminal contact network

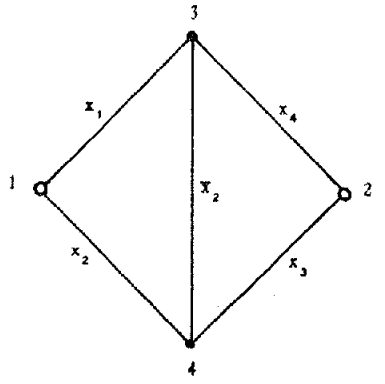


FIG. 5. Weighted graph corresponding to Fig. 4

of Figure 4. We associate with this network a weighted graph (Fig. 5) in an obvious way. The branch weights of this graph will be considered as B-ternary functions. Clearly, the set L_n^B of B-ternary n-variable functions is closed under the operations of addition, multiplication and inversion. Also all the algebraic rules given in Section 2 are satisfied. Thus L_n^B forms a distributive lattice with the constant functions 0 and 1 as zero and universal elements, respectively. The weighted graph of Figure 5 may thus be considered as *switching net*, defined in [14] as finite graph,[3] the branches of which are weighted by elements of a distributive lattice with 0 and 1. Thus, the theory of switching nets [14] applies.

First, we introduce the concept of *ternary transmission function* of a given two-terminal contact network. For the network of Figure 4 this is the B-ternary function

$$f = x_1 \cdot x_4 + x_1 \cdot \bar{x}_2 \cdot x_3 + x_2 \cdot x_3 + x_2 \cdot \bar{x}_2 \cdot x_4 . \tag{10}$$

Formally, the ternary transmission function of a two-terminal contact network s obtained from its switching net as follows:

1. Determine all irredundant input-output paths (tie sets).
2. For each such path obtain its *weight* by multiplying the corresponding branch weights.
3. Form the sum of all path weights obtained in 2.

The dual procedure to obtain the ternary transmission function of a two-terminal contact network from its switching net is as follows:

1. Determine all minimal cut-sets separating the two terminals.
2. For each such cut-set obtain its *weight* by adding the corresponding branch weights.
3. Form the product of all cut-set weights obtained in 2.

[3] For the purposes of this study we consider the graph to be non-oriented.

Applied to Figure 4, this dual procedure leads to

$$f = (x_1 + x_2)\cdot(x_1 + \bar{x}_2 + x_3)\cdot(x_2 + \bar{x}_2 + x_4)\cdot(x_3 + x_4). \tag{11}$$

By applying the distributive and absorption laws one easily derives (11) from (10), and vice versa. Alternatively, the equivalence between (10) and (11) can be shown by obtaining the corresponding ternary tables of combinations.

Other methods to derive the ternary transmission function of a two-terminal contact network are based on a suitable generalization of Boolean matrix theory [14, 15]. E.g., Figure 6 shows the (primitive) connection matrix C corresponding to Fig. 4 [15, 16]. The B-ternary transmission function f for Fig. 4 is obtained from C by first deleting column 1 and row 2 and then evaluating the (ternary) lattice-determinant.

Thus,

$$f = \begin{vmatrix} 0 & x_1 & x_2 \\ x_4 & 1 & \bar{x}_2 \\ x_3 & \bar{x}_2 & 1 \end{vmatrix} = x_1\cdot(x_4 + \bar{x}_2\cdot x_3) + x_2\cdot(x_3 + \bar{x}_2\cdot x_4)$$

which is the same ternary function as (10).

The B-ternary function (10) completely describes the steady-state and transient behavior of the network of Figure 4. Hazard detection is thus again possible by the application of Theorems 2 and 3, or their duals.

The B-ternary (transmission) function of a series-parallel contact network may be derived either by the method of Section 2 or by the methods outlined above. The equivalence of the results obtained is easily verified.

It is expected that the above formalism will also facilitate the adaptation of available techniques for the synthesis of binary non-series-parallel networks, e.g., Gould's method [17], to the ternary case. Methods for the synthesis of minimal hazard-free non-series-parallel contact networks can thus be obtained. Further research in this direction is required.

	1	2	3	4
1	1	0	x_1	x_2
2	0	1	x_4	x_3
3	x_1	x_4	1	$\bar{x}_2$
4	x_2	x_3	$\bar{x}_2$	1

FIG. 6. Primitive connection matrix for network of Fig. 4

wx \ yz	00	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	0	1	1
10	1	0	0	1

FIG. 7. Karnaugh map of binary transmission function F

8. *Application of Contact Sequences to Hazard Prevention*

Methods for hazard prevention in contact networks by the utilization of continuity transfer (make-before-break) contacts are well known [7]. A formal and more general approach to such techniques will be presented in this section. A relay spring pile-up is usually constructed in such a way as to ensure simultaneous action of all contacts in the pile-up. However, by obvious mechanical arrangements the contact next to the armature can be made to operate early, i.e. before any of the other contacts on the relay. Similarly, late contact action is feasible; i.e. one of the contacts will operate subsequent to all other contact operations. Also, early-acting contacts will release last, late-acting contacts first. Early- and late-acting contacts are sometimes referred to as "X-type" and "Y-type" contacts, respectively [18]. The performance of early- and late-acting contacts are represented by ternary transmission functions as shown in Table 2.

The application of such contacts considerably facilitates the design of hazard-free contact networks. As an example consider the binary transmission function

TABLE 2. TERNARY TRANSMISSION FUNCTIONS REPRESENTING EARLY- AND LATE-OPERATING CONTACTS

Type of Contact		*Relay position x* 0	$\frac{1}{2}$	1	*Corresponding ternary function*
early-operating	make	0	1	1	$\overline{x^0}$
	break	1	0	0	x^0
late-operating	make	0	0	1	x^1
	break	1	1	0	$\overline{x^1}$

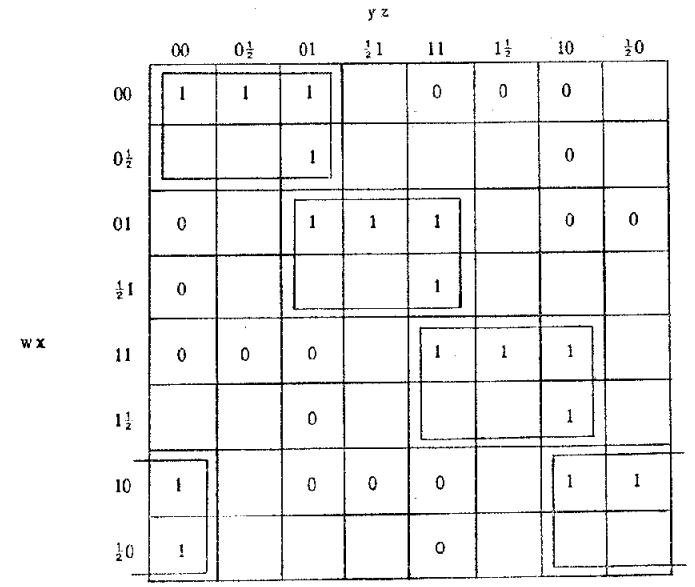


FIG. 8. Ternary map extension of Fig. 7

F defined by the Karnaugh map of Figure 7. We wish to design a hazard-free two-terminal contact network with F as its steady-state transmission function. First, we extend the map of Figure 7 into a ternary map (Fig. 8) indicating all function values to be realized by the network. The rectangular groupings indicated in Figure 8 immediately lead to the following ternary function g satisfying the requirements of Fig. 8:

$$g = w^0 \cdot \overline{x^1} \cdot y^0 + \overline{w^1} \cdot x^1 \cdot z^1 + w^1 \cdot \overline{x^0} \cdot y^1 + \overline{w^0} \cdot x^0 \cdot z^0$$

A network realization of g is not practicable, as it would require all contacts to be of the early- or late-operating type. However, it is not difficult to verify that the following function f will also satisfy the requirements of Figure 8:

$$f = \bar{w} \cdot \overline{x^1} \cdot \bar{y} + \overline{w^1} \cdot x \cdot z + w \cdot \overline{x^0} \cdot y + \overline{w^0} \cdot \bar{x} \cdot \bar{z}.$$

The function f is obtained from g by replacing w^0, w^1, etc., if not inverted, by $w^{(0)} = \bar{w}$, $w^{(1)} = w$, etc. (compare with proof of Theorem 5). The function f can be realized by the contact-sequencing technique discussed in this section. On the other hand, many more contacts are required to eliminate hazards if the standard methods described in the previous sections are applied.

Conclusion

The applicability of ternary switching algebra to the study of static hazards in combinational switching networks has been demonstrated. This ternary algebra has been shown to be a natural extension of Boolean algebra, suitable for an adequate description of both the steady-state and the transient behavior of combinational networks. Recently, considerable interest has been shown in nonbinary switching circuits and the need for generalizing conventional switching theory has been pointed out.

It is hoped that this paper will contribute to familiarizing the switching engineer with nonbinary methods in general.

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